

AN UPPER BOUND FOR THE CONCORDANCE STABLE RANGE

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ABSTRACT. We prove that the concordance stable range for manifolds of dimension $2n$ can be no better than $2n - 2$. In order to do so, we define new characteristic classes for block bundles, extending earlier work with Ebert, and prove their non-triviality.

For a smooth manifold M , possibly with boundary, the space of smooth concordances is $C(M) := \text{Diff}(M \times [0, 1] \text{ rel } M \times \{0\})$. There is a canonical map

$$(0.1) \quad C(M) \longrightarrow C(M \times I)$$

given by crossing with the interval I (and smoothing corners), and the (smooth) *concordance stable range* is the function

$$\phi(n) := \max\{k \in \mathbb{N} \mid (0.1) \text{ is } k\text{-connected for all manifolds } M \text{ of dimension } \geq n\}.$$

The main theorem concerning this function is due to Igusa [12], and says that

$$\phi(n) \geq \min\{\frac{n-7}{2}, \frac{n-4}{3}\}.$$

In this note we establish the following upper bound for this function.

Theorem A. $\phi(2n) \leq 2n - 2$ as long as $2n \geq 6$.

To explain our approach, let $W_{g,1} := \#^g S^n \times S^n \setminus \text{int}(D^{2n})$ with $2n \geq 6$, and consider the fibration sequence

$$(0.2) \quad \frac{\widetilde{\text{Diff}}_{\partial}(W_{g,1})}{\text{Diff}_{\partial}(W_{g,1})} \longrightarrow B\text{Diff}_{\partial}(W_{g,1}) \xrightarrow{i} B\widetilde{\text{Diff}}_{\partial}(W_{g,1})$$

from the classifying space of the group of diffeomorphisms of $W_{g,1}$ to the classifying space of the group of block diffeomorphisms of $W_{g,1}$. The rational cohomology of $B\text{Diff}_{\partial}(W_{g,1})$ has been computed for $g \gg 0$ by Galatius and the author in [9, 8]; the rational cohomology of $B\widetilde{\text{Diff}}_{\partial}(W_{g,1})$ has been computed for $g \gg 0$ by Berglund and Madsen in [1, 2] and in forthcoming work. Ebert and the author have shown in [5] that the map i is surjective on rational cohomology in the stable range.

Our approach to Theorem A is motivated by forthcoming work of Berglund and Madsen, in which they show that the map i is injective in degrees $*$ $< 2n$ and $g \gg 0$, and more importantly for our current purpose they show that this is sharp, in the following sense.

Proposition B (Berglund–Madsen). *For $g \gg 0$,*

$$(0.3) \quad \text{Ker}(i^* : H^{2n}(B\widetilde{\text{Diff}}_{\partial}(W_{g,1}); \mathbb{Q}) \rightarrow H^{2n}(B\text{Diff}_{\partial}(W_{g,1}); \mathbb{Q})) \neq 0.$$

This has implications for the Serre spectral sequence of (0.2), and it is this that we shall exploit to prove Theorem A. As Proposition B is central to our argument, and its proof is not yet available, in Sections 2 and 3 we will give an independent proof of it, which works for all $g \geq 1$ and does not require the computation of both groups. It consists of defining Mumford–Morita–Miller classes for block bundles,

which extend those that we already defined with Ebert in [5], and then showing that a certain such class—namely $\tilde{\kappa}_{e^2} - \tilde{\kappa}_{p_n}$, which is easily seen to lie in the kernel (0.3)—is not trivial. The construction of these classes and their non-triviality may be of interest independently of Theorem A.

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1. PROOF OF THEOREM A

By the work of Weiss–Williams [18, Theorem A], there is a certain map

$$(1.1) \quad \frac{\widetilde{\text{Diff}}_{\partial}(W_{g,1})}{\text{Diff}_{\partial}(W_{g,1})} \longrightarrow \Omega^{\infty}(S_+^{\infty} \wedge_{\mathbb{Z}/2} \Omega \mathbf{Wh}_s^{\text{Diff}}(W_{g,1}))$$

which is $(\phi(2n)+1)$ -connected. The $(\mathbb{Z}/2)$ -spectrum $\mathbf{Wh}_s^{\text{Diff}}(W_{g,1})$ is the 1-connected cover of the (smooth) Whitehead spectrum $\mathbf{Wh}^{\text{Diff}}(W_{g,1})$, which in turn is related to Waldhausen’s algebraic K -theory of spaces by a (split) cofibre sequence of spectra

$$(1.2) \quad \Sigma_+^{\infty} W_{g,1} \longrightarrow \mathbf{A}(W_{g,1}) \longrightarrow \mathbf{Wh}^{\text{Diff}}(W_{g,1}).$$

This identification requires the stable parameterised h -cobordism theorem [16].

Our strategy is then as follows. We use a theorem of Hsiang–Staffeldt to compute the spectrum cohomology $H^*(\mathbf{Wh}^{\text{Diff}}(W_{g,1}); \mathbb{Q})$ in degrees $* \leq 2n$. We take care to compute this *as a representation of the mapping class group $\Gamma_{g,1}$ of $W_{g,1}$* , in terms of the standard representation

$$H_g := H_n(W_{g,1}; \mathbb{Q})$$

of $\Gamma_{g,1}$. The spectrum cohomology of $S_+^{\infty} \wedge_{\mathbb{Z}/2} \Omega \mathbf{Wh}_s^{\text{Diff}}(W_{g,1})$ is then given by truncating, desuspending, and taking $\mathbb{Z}/2$ -invariants, and the cohomology of $\Omega^{\infty}(S_+^{\infty} \wedge_{\mathbb{Z}/2} \Omega \mathbf{Wh}_s^{\text{Diff}}(W_{g,1}))$ is the free graded-commutative algebra on the result.

We now suppose for a contradiction that $\phi(2n) \geq 2n - 1$, so the map (1.1) is $2n$ -connected and hence we have a computation of the rational cohomology of $\frac{\widetilde{\text{Diff}}_{\partial}(W_{g,1})}{\text{Diff}_{\partial}(W_{g,1})}$ in degrees $* \leq 2n - 1$, as a $\Gamma_{g,1}$ -module. We then study the Serre spectral sequence for (0.2), and derive a contradiction.

1.1. Rational homology of the Whitehead spectrum. We shall use Corollary 1.2 of Hsiang–Staffeldt [11], which shows that

$$H_*(\mathbf{A}(W_{g,1}); \mathbb{Q}) = \pi_*(\mathbf{A}(W_{g,1})) \otimes \mathbb{Q} \cong (K_*(\mathbb{Z}) \otimes \mathbb{Q}) \oplus (\Sigma \bar{K}_{ab})$$

where K is a minimal model for the dga $C_*(\Omega W_{g,1}; \mathbb{Q})$, \bar{K} denotes the augmentation ideal, which inherits the structure of a graded Lie algebra with bracket

$$[x, y] := x \cdot y - (-1)^{|x| \cdot |y|} y \cdot x,$$

and $\bar{K}_{ab} = \bar{K}/[\bar{K}, \bar{K}]$ is the abelianisation of this graded Lie algebra.

As $W_{g,1}$ is a suspension, the homology of $\Omega W_{g,1}$ is the tensor algebra on the vector space $H_g[n-1]$. In particular it is a free (non-commutative) algebra, so is quasi-isomorphic to $C_*(\Omega W_{g,1}; \mathbb{Q})$, and we make take $K = H_*(\Omega W_{g,1}; \mathbb{Q})$ with trivial differential. It follows that \bar{K}_{ab} is the augmentation ideal of the free graded commutative algebra on $H_g[n-1]$, that is

$$\bar{K}_{ab} = (H_g[n-1]) \oplus \left(\begin{array}{l} \text{Sym}^2(H_g)[2n-2] \text{ if } n \text{ is odd} \\ \wedge^2(H_g)[2n-2] \text{ if } n \text{ is even} \end{array} \right) \oplus (\text{terms of degree } \geq 3n-3).$$

Let us write

$$U := \begin{pmatrix} \text{Sym}^2(H_g) & \text{if } n \text{ is odd} \\ \wedge^2(H_g) & \text{if } n \text{ is even} \end{pmatrix}.$$

Then we have

$$H_*(\mathbf{A}(W_{g,1}); \mathbb{Q}) \cong (K_*(\mathbb{Z}) \otimes \mathbb{Q}) \oplus (H_g[n]) \oplus (U[2n-2])$$

in degrees $* \leq 2n$. Applying the cofibre sequence (1.2), we obtain

$$H_*(\mathbf{Wh}^{\text{Diff}}(W_{g,1}); \mathbb{Q}) \cong (\tilde{K}_*(\mathbb{Z}) \otimes \mathbb{Q}) \oplus (U[2n-2])$$

in degrees $* \leq 2n$. The rational homology of $\mathbf{Wh}_s^{\text{Diff}}(W_{g,1})$ is therefore the same, as it is already 1-connected. Thus, dualising, we have

$$H^*(S_+^\infty \wedge_{\mathbb{Z}/2} \Omega \mathbf{Wh}_s^{\text{Diff}}(W_{g,1}); \mathbb{Q}) \cong ((\tilde{K}_{*-1}(\mathbb{Z}) \otimes \mathbb{Q}) \oplus (U[2n-2]))^{\mathbb{Z}/2}$$

in degrees $* \leq 2n-1$, for some involution. It follows from Farrell–Hsiang [6] (which considers the case $g=0$) that this involution acts as -1 on $\tilde{K}_{*-1}(\mathbb{Z}) \otimes \mathbb{Q}$, so this summand does not contribute to the invariants. Thus

$$H^*(S_+^\infty \wedge_{\mathbb{Z}/2} \Omega \mathbf{Wh}_s^{\text{Diff}}(W_{g,1}); \mathbb{Q}) \cong (U[2n-2])^{\mathbb{Z}/2}$$

in degrees $* \leq 2n-1$, for some involution on U . Taking the free graded-commutative algebra on this, it follows that

$$H^*(\Omega^\infty(S_+^\infty \wedge_{\mathbb{Z}/2} \Omega \mathbf{Wh}_s^{\text{Diff}}(W_{g,1})); \mathbb{Q}) \cong \mathbb{Q}[0] \oplus (U[2n-2])^{\mathbb{Z}/2}$$

in degrees $* \leq 2n-1$.

1.2. The Serre spectral sequence argument. The Serre spectral sequence for the fibration (0.2) takes the form

$$E_1^{p,q} = H^p(B\widetilde{\text{Diff}}_\partial(W_{g,1}); H^q\left(\frac{\widetilde{\text{Diff}}_\partial(W_{g,1})}{\text{Diff}_\partial(W_{g,1})}; \mathbb{Q}\right)) \Rightarrow H^{p+q}(B\text{Diff}_\partial(W_{g,1}); \mathbb{Q}).$$

Under the assumption that $\phi(2n) \geq 2n-1$ we have identified the coefficients in degrees $q \leq 2n-1$, to be \mathbb{Q} for $q=0$ and to be $V := U^{\mathbb{Z}/2}$ for $q=2n-2$. In order for (0.3) to be possible, we must therefore have a non-trivial differential

$$d^{2n-1} : H^1(B\widetilde{\text{Diff}}_\partial(W_{g,1}); V) \longrightarrow H^{2n}(B\widetilde{\text{Diff}}_\partial(W_{g,1}); \mathbb{Q}).$$

In particular, the source must be non-trivial. Note that

$$H^1(B\widetilde{\text{Diff}}_\partial(W_{g,1}); V) \cong H^1(B\widetilde{\text{Diff}}_\partial(W_{g,1}); U)^{\mathbb{Z}/2},$$

for some involution on U , so the following shall give a contradiction.

Proposition 1.1. $H^1(B\widetilde{\text{Diff}}_\partial(W_{g,1}); U) = 0$ for $g \gg 0$.

Proof. The action of $\Gamma_{g,1}$ on $H_n(W_{g,1}; \mathbb{Z})$ preserves the intersection form, determining a homomorphism

$$\Gamma_{g,1} \longrightarrow \begin{cases} O_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even} \\ Sp_{2g}(\mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases}$$

This is onto if n is even, but not if n is odd (unless $n=1, 3, 7$) cf. [2, Example 4.2]. Let A_g denote its image, which is always an arithmetic subgroup and Zariski dense in $O_{g,g}(\mathbb{C})$ or $Sp_{2g}(\mathbb{C})$. Consider the fibration sequence

$$B\widetilde{\text{Tor}}_{g,1} \longrightarrow B\widetilde{\text{Diff}}_\partial(W_{g,1}) \longrightarrow BA_g,$$

where $B\widetilde{\text{Tor}}_{g,1}$ is defined to be the homotopy fibre. By [2, Proposition 4.1] we have

$$H^1(B\widetilde{\text{Tor}}_{g,1}; \mathbb{Q}) \cong \begin{cases} H_g & n \equiv 3 \pmod{4} \\ 0 & \text{else} \end{cases}$$

so if $n \not\equiv 3 \pmod{4}$ then $H^1(A_g; U) \rightarrow H^1(\widetilde{BDiff}_\partial(W_{g,1}); U)$ is an isomorphism, and if $n \equiv 3 \pmod{4}$ then we have an exact sequence

$$0 \longrightarrow H^1(A_g; U) \longrightarrow H^1(\widetilde{BDiff}_\partial(W_{g,1}); U) \longrightarrow (H_g \otimes U)^{A_g}.$$

In the case $n \equiv 3 \pmod{4}$, n is odd and Zariski density implies that the complexification of $(H_g \otimes \text{Sym}^2(H_g))^{A_g}$ is $(H_g \otimes \text{Sym}^2(H_g) \otimes \mathbb{C})^{Sp_{2g}(\mathbb{C})}$, which is contained in $(H_g^{\otimes 3} \otimes \mathbb{C})^{Sp_{2g}(\mathbb{C})}$, which vanishes by standard invariant theory (for which we refer to [7, §F.2]).

It remains to show that $H^1(A_g; U) = 0$. The representation U is arithmetic, so a theorem of Borel [3, Theorem 1] identifies this with $H^1(A_g; \mathbb{Q}) \otimes U^{A_g}$ for $g \gg 0$, so it is enough to show the vanishing of U^{A_g} . If n is odd then U^{A_g} is $\text{Sym}^2(H_g)^{A_g}$, whose complexification is the same as $\text{Sym}^2(H_g \otimes \mathbb{C})^{Sp_{2g}(\mathbb{C})}$ by Zariski density, and this vanishes by standard invariant theory. If n is even then U^{A_g} is $\wedge^2(H_g)^{A_g}$, whose complexification is $\wedge^2(H_g \otimes \mathbb{C})^{O_{g,g}(\mathbb{C})}$, which also vanishes by standard invariant theory (noting that $O_{g,g}(\mathbb{C}) \cong O_{2g}(\mathbb{C})$). \square

2. CHARACTERISTIC CLASSES OF BLOCK BUNDLES

We should like to give a proof of Proposition B, as it does not require the entire corpus [9, 8, 1, 2] and beyond to see that the kernel (0.3) is non-trivial. We shall show that this kernel is non-trivial by producing an explicit element in it, which will be described in terms of generalised Mumford–Morita–Miller classes. If $(\pi : E \rightarrow |K|, \mathcal{A})$ is a smooth oriented block bundle with fibre a closed d -manifold M (we refer to [5, Section 2] for this notation), in [5, Section 3] Ebert and the author have associated to it

- (i) a Leray–Serre spectral sequence $H^p(|K|, \mathcal{H}^q(M)) \Rightarrow H^{p+q}(E)$, and hence a fibre-integration map $\pi_!(-) : H^{k+d}(E) \rightarrow H^k(|K|)$,
- (ii) a transfer map $\text{trf}_\pi^* : H^*(E) \rightarrow H^*(|K|)$ of Becker–Gottlieb type,
- (iii) a *stable* vertical tangent bundle $T_\pi^s E \rightarrow E$,

such that if $(\pi : E \rightarrow |K|, \mathcal{A})$ arises from a smooth fibre bundle then these data reduce to those coming from the bundle structure. In the case $d = 2n$, we then employed the following ruse: If π came from a smooth fibre bundle with $2n$ -dimensional fibres, so there was an *unstable* vertical tangent bundle $T_\pi E$, then we would have $e(T_\pi E)^2 = p_n(T_\pi E)$, and $\pi_!(e(T_\pi E) \cdot -) = \text{trf}_\pi^*(-) : H^*(E) \rightarrow H^*(|K|)$. Therefore, for a monomial p_I in Pontrjagin classes, if we define

$$\tilde{\kappa}_{p_I}(\pi) := \pi_!(p_I(T_\pi^s E)) \quad \tilde{\kappa}_{ep_I} := \text{trf}_\pi^*(p_I(T_\pi^s E))$$

then these classes restrict to the usual κ_{p_I} and κ_{ep_I} on fibre bundles, and these give all generalised Mumford–Morita–Miller classes on fibre bundles.

By way of apology for this ruse, we add to the list above

- (iv) an Euler class $e(T_\pi E) \in H^d(E; \mathbb{Z})$.

(Of course $e(T_\pi E)$ is merely notation: there is no d -dimensional bundle $T_\pi E$ of which it is the Euler class.) Using this Euler class, we may then define

$$\tilde{\kappa}_{e^i p_I}(\pi) := \pi_!(e(T_\pi E)^i \cdot p_I(T_\pi^s E)) \in H^*(|K|; \mathbb{Z}).$$

The symbol $\tilde{\kappa}_{ep_I}$ has the same meaning as before, by Lemma 2.2 (iv) below.

The existence of this Euler class is a consequence of the Fibre Inclusion Theorem of [4] (or rather its proof, which constructs a canonical such class), and the fact that the homotopy fibre of π is homotopy equivalent to a Poincaré duality space of dimension d , namely M [5, Proposition 2.8]. As the construction is quite pretty, let us describe it.

Construction 2.1. Embed $|K|$ into \mathbb{R}^k for some $k \gg 0$, and let B' be a closed regular neighbourhood, so that there is a retraction $r : B' \rightarrow |K|$. Let $B = D(B')$ be the double of B' , a closed smooth manifold. This has a retraction $s : D(B') \rightarrow B'$, and let $p : X \rightarrow B$ be the Hurewicz fibration obtained by turning π into a fibration $\pi^f : E^f \rightarrow |K|$ and pulling it back along rs . As B and the fibre of p are Poincaré duality spaces, of dimensions k and d respectively, X is too [10], of dimension $(d+k)$. But $X \times_B X = p^*(X) \rightarrow X$ is also a fibration over a Poincaré duality space with Poincaré duality fibre, so is again a Poincaré duality space, of dimension $(2d+k)$. Writing $\Delta : X \rightarrow X \times_B X$ for the fibrewise diagonal map, which admits an umkehr map $\Delta_!$ as source and target are both Poincaré, we define

$$e(T_p X) := \Delta^* \Delta_!(1) \in H^d(X; \mathbb{Z}).$$

We then define $e(T_\pi E)$ by restriction along $E \subset E^f \subset X|_{B'} \subset X$.

It is easy to see that the class so obtained is independent of all choices, and it is shown in [4, §4] that it restricts to the Euler class on the fibre M . The definition given in [4, §4] seems to differ by a sign, but it does not, by Lemma 2.2 (i) below.

Lemma 2.2. *The Euler class defined enjoys the following properties:*

- (i) *If d is odd then $2e(T_\pi E) = 0 \in H^*(E; \mathbb{Z})$,*
- (ii) *if $(\pi : E \rightarrow |K|, \mathcal{A})$ arises from a smooth fibre bundle with vertical tangent bundle $T_\pi E$, then $e(T_\pi E)$ agrees with the Euler class of the vertical tangent bundle,*
- (iii) *if there is a map $r : E \rightarrow M$ such that $\pi \times r : E \rightarrow |K| \times M$ is a homotopy equivalence, then $e(T_\pi E) = r^*(e(TM))$,*
- (iv) *the equation $\pi_!(e(T_\pi E) \cdot -) = \text{trf}_\pi^*(-) : H^*(E; \mathbb{Z}) \rightarrow H^*(|K|; \mathbb{Z})$ is satisfied.*

Proof. For (i), consider the involution τ of $X \times_B X$ which interchanges the two factors. When d is odd, this has degree -1 , and so $\tau^* \Delta_! = -\Delta_!$. On the other hand $\Delta^* \tau^* = \Delta^*$, so $e(T_\pi E) = -e(T_\pi E)$.

For (ii), note that if $(\pi : E \rightarrow |K|, \mathcal{A})$ arises from a smooth fibre bundle then in Construction 2.1 we do not need to replace it by a fibration. The resulting $p : X \rightarrow B$ is a smooth fibre bundle with vertical tangent bundle $T_p X$, and the map $\Delta : X \rightarrow X \times_B X$ is a smooth embedding with normal bundle $T_p X$. Hence $\Delta^* \Delta_!(1)$ is the Euler class of $T_p X$, which restricts to the Euler class of $T_\pi E$.

For (iii), if such an r exists then the fibration $p : X \rightarrow B$ admits a similar fibre homotopy trivialisation, $p \times \rho : X \xrightarrow{\sim} B \times M$. Then $X \times_B X \simeq B \times M \times M$ and the map Δ is given by the identity map on B and the diagonal map on M . Hence $\Delta^* \Delta_!(1) = 1 \otimes e(TM)$.

For (iv), we must involve ourselves in the details of the construction of the transfer in [4], with which we assume the reader is familiar. We begin by constructing a commutative diagram

$$\begin{array}{ccccccc}
 & & F & \xleftarrow{\quad} & W \times T^k & \xleftarrow{\quad} & D(W) \times T^k \\
 & & \downarrow & & \downarrow & & \downarrow \\
 E & \xleftarrow{r} & X & \xleftarrow{t} & X' & \xleftarrow{s} & X'' \\
 \downarrow \pi & & \downarrow p & \xleftarrow{u} & \downarrow p' & \xleftarrow{v} & \downarrow p'' \\
 |K| & \xleftarrow{\quad} & B & \xleftarrow{\quad} & B & \xleftarrow{\quad} & B
 \end{array}$$

In this diagram, B is a Poincaré duality space and p is a Hurewicz fibration with fibre $F \simeq M^d$ (obtained as in Construction 2.1). W is a smooth oriented manifold of dimension $(d+\ell)$ with boundary, which is homotopy equivalent to M , and p' is a smooth fibre bundle (obtained from the Closed Fibre Smoothing Theorem of

[4]). The map p'' is obtained as the fibrewise double of p' , and is a smooth oriented fibre bundle with closed fibres. Finally, the horizontal arrows express each left-hand space as a (fibrewise) retract of the the right-hand space.

For a fibration $p : S \rightarrow T$ with fibre homotopy equivalent to a finite CW complex, and a fibrewise map $f : S \rightarrow S$, let us write $\text{trf}_{p,f}^* : H^*(S) \rightarrow H^*(T)$ for the associated transfer map. This is the map denoted τ^f in [4]. When $f = \text{Id}_S$, we shorten this to trf_p^* .

By the definition of the transfer in [4, §6], we have $\text{trf}_p^* = \text{trf}_{p'',vuts}^* s^* t^*$. By the construction of the transfer for smooth fibre bundles in [4, §5], if we write

$$\begin{aligned} \delta &= (\text{Id}_{X''}, vuts) : X'' \longrightarrow X'' \times_B X'' \\ d &= (\text{Id}_{X''}, \text{Id}_{X''}) : X'' \longrightarrow X'' \times_B X'' \end{aligned}$$

then we have $\text{trf}_{p'',vuts}^*(-) = p'_!(\delta^*(d_!(1)) \cdot -)$. Thus the map $\text{trf}_p^*(-)$ is $p'_!(\delta^*(d_!(1)) \cdot s^* t^*(-)) = (pts)_!(\delta^*(d_!(1)) \cdot s^* t^*(-))$, which we may write as $p_!((ts)_!(\delta^*(d_!(1)) \cdot -))$, so we will be done if $(ts)_!(\delta^*(d_!(1)))$ is equal to the class $e(T_p X)$ defined by Construction 2.1. Consider the homotopy pullback squares

$$\begin{array}{ccc} X'' & \xrightarrow{ts} & X \\ \downarrow \delta & & \downarrow (\text{Id} \times vu) \circ \Delta \\ X'' \times_B X'' & \xrightarrow{ts \times \text{Id}} & X \times_B X'' \end{array} \quad \begin{array}{ccc} X'' & \xrightarrow{(ts \times \text{Id}) \circ d} & X \times_B X'' \\ \downarrow ts & & \downarrow \text{Id} \times ts \\ X & \xrightarrow{\Delta} & X \times_B X \end{array}$$

of Poincaré duality spaces, to which Lemma 2.3 below applies and shows that

$$(ts)_! \delta^* = \Delta^* (\text{Id} \times vu)^* (ts \times \text{Id})_! \quad (ts \times \text{Id})_! d_! (ts)^* = (\text{Id} \times ts)^* \Delta_!.$$

(The signs can be determined by restricting each square to a single fibre over B .)

Thus, writing $1 = (ts)^*(1)$, we have

$$\begin{aligned} (ts)_! \delta^* d_! (ts)^*(1) &= \Delta^* (\text{Id} \times vu)^* (ts \times \text{Id})_! d_! (ts)^*(1) \\ &= \Delta^* (\text{Id} \times vu)^* (\text{Id} \times ts)^* \Delta_! (1) = \Delta^* \Delta_! (1) \end{aligned}$$

which is $e(T_p X)$, as required. \square

Lemma 2.3. *Consider a homotopy cartesian square*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow u & & \downarrow v \\ B & \xrightarrow{f} & D \end{array}$$

of oriented Poincaré duality spaces. Then $g_! u^ = \pm v^* f_!$.*

The sign ambiguity is unavoidable under the given hypotheses: changing the orientation of B , say, does not change $g_! u^*$, but changes $v^* f_!$ by a sign.

Proof. Let us write a for the formal dimension of A , and so on. We assume some familiarity with the notion of Poincaré embeddings, for which we refer to [13] for details. It is enough to prove the identity for the larger square

$$\begin{array}{ccccc} A & \xrightarrow{g} & C & \xrightarrow{\text{Id}_C \times \{*\}} & C \times S^N \\ \downarrow u & & \downarrow v & & \downarrow v \times \text{Id}_{S^N} \\ B & \xrightarrow{f} & D & \xrightarrow{\text{Id}_D \times \{*\}} & D \times S^N. \end{array}$$

By this device, we may suppose [13, Lemma 3.1] that f admits the structure of a Poincaré embedding, with complement K and normal spherical fibration ξ of dimension $(d - b - 1)$. Let $u^* \xi \rightarrow A$ denote the pulled back spherical fibration, and

$v^*K \rightarrow C$ denote the homotopy pullback of the map $K \rightarrow D$ along v . There is then a homotopy commutative cube

$$\begin{array}{ccccc}
 u^*\xi & \longrightarrow & v^*K & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & A & \longrightarrow & C \\
 & & \downarrow & & \downarrow \\
 \xi & \longrightarrow & K & & \\
 & \searrow & \downarrow & \searrow & \\
 & & B & \longrightarrow & D
 \end{array}$$

in which the bottom face is homotopy cocartesian, and the vertical faces are all homotopy cartesian. It follows by Mather's Second Cube Theorem [14, Theorem 25] that the top face is also homotopy cocartesian. We therefore have a map

$$C \simeq A \cup_{u^*\xi} v^*K \longrightarrow A/u^*\xi = \text{Th}(u^*\xi)$$

by collapsing v^*K , and similarly for K . This gives a homotopy commutative diagram

$$\begin{array}{ccc}
 C & \longrightarrow & \text{Th}(u^*\xi \rightarrow A) \\
 \downarrow v & & \downarrow \text{Th}(u) \\
 D & \longrightarrow & \text{Th}(\xi \rightarrow B)
 \end{array}$$

which in cohomology yields the required equation. From this point of view, the sign ambiguity arises from the two possible choices of Thom class for $u^*\xi$: the one compatible with $[C]$ and $[A]$, or the pullback of the one compatible with $[D]$ and $[B]$. \square

3. PROOF OF PROPOSITION B

We can extend the definition of the classes $\tilde{\kappa}_{e^i p_I}$ to block bundles having fibres $W_{g,1}$ by filling in a disc in each fibre, giving a new block bundle with fibre $W_g := W_{g,1} \cup_{\partial} D^{2n} = \#^g S^n \times S^n$. There are therefore defined universal characteristic classes $\tilde{\kappa}_{e^i p_I} \in H^*(B\widetilde{\text{Diff}}_{\partial}(W_{g,1}); \mathbb{Q})$, by the proof of [5, Theorem 3.4].

In particular, we have a class $\tilde{\kappa}_{e^2} - \tilde{\kappa}_{p_n} \in H^{2n}(B\widetilde{\text{Diff}}_{\partial}(W_{g,1}); \mathbb{Q})$ which vanishes in $H^{2n}(B\widetilde{\text{Diff}}_{\partial}(W_{g,1}); \mathbb{Q})$, because $e^2 = p_n$ on the total space of a smooth fibre bundle. Proposition B is an immediate consequence of the following.

Proposition 3.1. *For each $g \geq 1$ and each $n \geq 3$ there is a block bundle $(\pi : E \rightarrow |K|, \mathcal{A})$ with fibre $W_{g,1}$, such that*

- (i) $\tilde{\kappa}_{e^2}(\pi) = 0 \in H^{2n}(|K|; \mathbb{Q})$,
- (ii) $\tilde{\kappa}_{p_n}(\pi) \neq 0 \in H^{2n}(|K|; \mathbb{Q})$.

Therefore $\tilde{\kappa}_{e^2} - \tilde{\kappa}_{p_n} \neq 0 \in H^{2n}(B\widetilde{\text{Diff}}_{\partial}(W_{g,1}); \mathbb{Q})$.

Proof. From Lemma 2.2 (iii) it follows that the $\tilde{\kappa}_{e^i}$ vanish for all $i > 0$ on all fibre homotopically trivial block bundles. We will therefore construct π to be fibre homotopically trivial, guaranteeing that $\tilde{\kappa}_{e^2}(\pi) = 0$.

We will use the (space-level) surgery fibration of Quinn [15], which following the discussion in [1, Section 3.2], in particular equation (43), may be put in the form

$$\left(\frac{\text{hAut}_{\partial}(W_{g,1})}{\widetilde{\text{Diff}}_{\partial}(W_{g,1})} \right)_{(1)} \longrightarrow \text{map}_*(W_{g,1}/\partial W_{g,1}, G/O)_{(1)} \xrightarrow{\sigma} \mathbb{L}_{2n}(\mathbb{Z})_{(1)}.$$

Thus to construct a fibre homotopically trivial block bundle over B (with some triangulation) it is enough to give a map $f : B \rightarrow \text{map}_*(W_{g,1}/\partial W_{g,1}, G/O)_{(1)}$ and a nullhomotopy of $\sigma \circ f$.

For simplicity of exposition we restrict to the case $n = 2k$. We let $B = S^n \times S^n$, write $a, b \in H^n(B; \mathbb{Q})$ for a hyperbolic basis, and write $e_1, f_1, \dots, e_g, f_g \in H^n(W_{g,1}, \partial W_{g,1}; \mathbb{Q})$ for a hyperbolic basis. Write the n th Hirzebruch L -polynomial as $\mathcal{L}_n = Ap_n + Bp_{n/2}^2$ modulo other Pontrjagin classes, for some constants A and B . It is well-known that $A \neq 0$, and less well-known but true [17, Lemma A.1] that $B \neq 0$.

As the composition

$$p : G/O \xrightarrow{i} BO \xrightarrow{\prod p_i} \prod_{i=1}^{\infty} K(\mathbb{Z}, 4i)$$

has homotopy fibre with finite homotopy groups, we claim that may find a map f whose adjoint $\hat{f} : (B \times W_{g,1}, B \times \partial W_{g,1}) \rightarrow (G/O, *)$ composed with i gives a class

$$\xi \in KO^0(B \times W_{g,1}, B \times \partial W_{g,1})$$

which has $p_{n/2}(\xi) = C \cdot (a \otimes e_1 + b \otimes f_1)$, $p_n(\xi) = -\frac{2BC^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1$, and all other rational Pontrjagin classes zero, for some constant $C \neq 0$. To establish this claim, let the map

$$\varphi : (B \times W_{g,1}, B \times \partial W_{g,1}) \longrightarrow \left(\prod_{i=1}^{\infty} K(\mathbb{Z}, 4i), * \right)$$

classify the pair of relative cohomology classes $L \cdot (a \otimes e_1 + b \otimes f_1)$ and $-\frac{2BL^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1$, for some integer $L \neq 0$ large enough that these classes are integral. For each $N > 0$ consider the map $\phi_N : \prod_i K(\mathbb{Z}, 4i) \rightarrow \prod_i K(\mathbb{Z}, 4i)$ which multiplies by N^i on $K(\mathbb{Z}, 4i)$. The diagram

$$\begin{array}{ccc} B \times \partial W_{g,1} & \xrightarrow{\varphi|_{B \times \partial W_{g,1}}} & * \\ \downarrow & \searrow \hat{f} & \downarrow \\ B \times W_{g,1} & \xrightarrow{\varphi} \prod_{i=1}^{\infty} K(\mathbb{Z}, 4i) & \xrightarrow{\phi_N} \prod_{i=1}^{\infty} K(\mathbb{Z}, 4i) \\ & & \downarrow p \\ & & G/O \end{array}$$

then admits a dotted lift \hat{f} for N large enough, as the universal obstructions to finding such a lift lie in the cohomology of $\prod_i K(\mathbb{Z}, 4i)$ with finite coefficients, and are therefore annihilated (on each skeleton) by some ϕ_N . The resulting map \hat{f} gives $p_{n/2}(\xi) = L \cdot N^{n/2} \cdot (a \otimes e_1 + b \otimes f_1)$, $p_n(\xi) = -\frac{2BL^2}{A} \cdot N^n \cdot a \cdot b \otimes e_1 \cdot f_1$, and all other Pontrjagin classes zero, as required (with $C = L \cdot N^{n/2}$).

We must show that the composition

$$B = S^n \times S^n \xrightarrow{f} \text{map}_*(W_{g,1}/\partial W_{g,1}, G/O)_{(1)} \xrightarrow{\sigma} \mathbb{L}_{2n}(\mathbb{Z})_{(1)}$$

is nullhomotopic, but we shall allow ourselves to precompose f with self-maps $k_N : S^n \times S^n \rightarrow S^n \times S^n$ having degree $N \neq 0$ on both factors (such a precomposition preserves the form of Pontrjagin classes which has been arranged above). With this in mind, it is enough to show that

$$\sigma \circ f = 0 \in [B, \mathbb{L}_{2n}(\mathbb{Z})] \otimes \mathbb{Q}.$$

This group may be identified with $H^{4*}(B; \mathbb{Q})$. If $n \equiv 0 \pmod{4}$ then the component of degree $n = 2k = 4\ell$ is identified with the Künneth factor of

$$\frac{1}{8} \mathcal{L}_{3\ell}(\xi) \in H^{12\ell}(B \times W_{g,1}, B \times \partial W_{g,1}; \mathbb{Q})$$

in $H^{4\ell}(B; \mathbb{Q}) \cong H^{4\ell}(B; \mathbb{Q}) \otimes H^{8\ell}(W_{g,1}, \partial W_{g,1}; \mathbb{Q})$. But $\mathcal{L}_{3\ell}(\xi) = 0$ by observation, as only $p_{4\ell}(\xi)$ and $p_{2\ell}(\xi)$ are non-zero. Whatever the class of n modulo 4, the component of degree $2n = 4k$ is identified with the Künneth factor of

$$\frac{1}{8}\mathcal{L}_{2k}(\xi) \in H^{8k}(B \times W_{g,1}, B \times \partial W_{g,1}; \mathbb{Q})$$

in $H^{4k}(B; \mathbb{Q})$. But by construction

$$\mathcal{L}_{2k}(\xi) = A \cdot \left(-\frac{2BC^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1\right) + B \cdot (C \cdot (a \otimes e_1 + b \otimes f_1))^2 = 0.$$

We therefore obtain a map f , with $\sigma \circ f$ nullhomotopic and $i \circ \hat{f}$ classifying a vector bundle ξ' having $p_{n/2}(\xi') = D \cdot (a \otimes e_1 + b \otimes f_1)$, $p_n(\xi') = -\frac{2BD^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1$, and all other Pontrjagin classes zero, for some constant $D \neq 0$. (The constant will have changed when we precomposed the original choice of f with the maps k_N .) The associated block bundle $\pi : E \rightarrow |K| \approx B$ has $T_v^s E \simeq_s TE - \pi^*TB = \epsilon^{2n} + \xi'$ (see [5, Lemma 3.3]) and so

$$\tilde{\kappa}_{p_n}(\pi) = \pi_!(p_n(T_v^s E)) = \pi_!(p_n(\xi')) = -\frac{2BD^2}{A} \cdot a \cdot b \neq 0$$

as required.

It is not difficult to adapt the above argument to work for $n = 2k + 1$. The essential point is that if we write $\mathcal{L}_n = Ap_n + Bp_{\frac{n-1}{2}}p_{\frac{n+1}{2}}$ modulo all other Pontrjagin classes, then $A \neq 0$ and again by [17, Lemma A.1] $B \neq 0$. We then take $B = S^{2k-1} \times S^{2k+3}$ and proceed as above. \square

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